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Hybrid inclusion and disclusion systems with applications to equilibria and parametric optimization

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Abstract In this paper, we first establish the existence theorems of the solution of hybrid inclusion and disclusion systems, from which we study mixed types of systems of generalized quasivariational inclusion and disclusion problems and systems of generalized vector quasi-equilibrium problems. Some applications of existence theorems to feasible points for various mathematical programs with variational constraints or equilibrium constraints, system of vector saddle point and system of minimax theorem are also given.

Keywords Hybrid inclusion and disclusion system · Equilibrium problem · Lin and Du's variant of Ekeland's variational principle · Quasivariational inclusion and disclusion problem · Vector saddle point theorem · Minimax theorem

1 Introduction

Let X be a nonempty subset of a topological space (t.s., for short) and $f : X \times X \to \mathbb{R}$ a function with $f(x, x) \geq 0$ for all $x \in X$. Then the scalar equilibrium problem (*EP*, for short) [5] is to find $\bar{x} \in X$ such that $f(\bar{x}, y) \geq 0$ for all $y \in X$. The equilibrium problem was extensively investigated and generalized to the vector equilibrium problems for single-valued or multi-valued maps and contains optimization problems, variational inequalities problems, saddle point problems, the Nash equilibrium problems, fixed point problems, complementary problems, bilevel problems and semi-infinite problems as special cases and have some applications in mathematical program with equilibrium constraint; for detail one can refer to [1,2,6,7,9–21,23] and references therein.

In 1979, Rubinov [22] studied the following variational inclusions problem (R):

$$\text{Given } x \in \mathbb{R}^n, \text{ find } y \in \mathbb{R}^m \text{ such that } 0 \in g(x, y) + Q(x, y), \qquad (R)$$

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where $g : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^p$ is a single-valued function and $Q : \mathbb{R}^n \times \mathbb{R}^m \multimap \mathbb{R}^p$ is a multivalued map. It is known that model (R) covers variational inequalities problems and a number of variational system to many applications. Since then various types of variational inclusions problems have been extended and generalized by Adly [1], Ahmad et al. [2], Ding [8] and Huang [10], etc. Recently, Mordukhovich [21] discussed the following problem:

$$\min_{(x,y)} \varphi(x, y), \text{ subject to } y \in S(x), x \in X,$$

where $S : X \multimap Y$ is given by $S(x) = \{y \in Y : 0 \in g(x, y) + Q(x, y)\}$ and $X \subseteq \mathbb{R}^n$, $Y \subseteq \mathbb{R}^m$ and $\varphi : X \times Y \to \mathbb{R}^s$, and also study the optimal conditions of this type of problem.

Recently, Lin [14,15,18,19] studied the existence theorems of systems of generalized variational inclusion and disclusion problems. By these results, he established some existence theorems of solutions of nonlinear problems; e.g. systems of generalized vector quasiequilibrium problem, collective variational fixed point, systems of generalized quasi-loose saddle point, systems of minimax theorem, mathematical program with systems of variational inclusions constraints, mathematical program with systems of equilibrium constraints, etc.

Motivated and inspired by the works mentioned above, in this paper we shall introduce and investigate the following new problem. Let I be any index set. For each $i \in I$, let Y_i be a nonempty closed convex subset of a Hausdorff topological vector space (t.v.s., for short) V_i, $H_i \subseteq Y_i$, $Y = \prod_{i \in I} Y_i$, $A_i : Y \multimap Y_i$ and $T_i : Y \multimap Y_i$ multivalued maps. The mathematical model about hybrid inclusion and disclusion systems (HIDS, for short) is defined as follows:

(HIDS) Find $v = (v)_{i \in I} \in Y$ such that $v_i \in H_i$ and
$y_i \notin A_i(v)$ for all $y_i \in T_i(v)$ and for all $i \in I$.

In fact, HIDS contains several important problems as special cases. We first give some examples in this section to interpret our idea and the usefulness of the theory and then explain how they correlate to some applications (see Sects. 3, 4).

Let X be a nonempty subset of a topological space E and $u \in X$ be given. For each $i \in I$, let U_i and Z_i be real t.v.s. with zero vectors θ_{U_i} and θ_{Z_i}, respectively.

Example (A) For each $i \in I$, let $F_i : X \times Y_i \multimap U_i$ and $G_i : X \times Y \times Y_i \multimap Z_i$ be multivalued maps with nonempty values. If H_i and A_i are defined as follows:

$$H_i = \{y_i \in Y_i : \theta_{U_i} \notin F_i(u, y_i)\}$$

and

$$A_i(y) = \{z_i \in Y_i : \theta_{Z_i} \notin G_i(u, y, z_i)\},$$

then HIDS will reduce to the following system of mixed type of parametric variational inclusion and disclusion problem (P_1):

(P_1) Find $v = (v)_{i \in I} \in Y$ such that $\theta_{U_i} \notin F_i(u, v_i)$ and $\theta_{Z_i} \in G_i(u, v, y_i)$ for all $y_i \in T_i(v)$ and for all $i \in I$.

Example (B) For each $i \in I$, let $F_i : X \times Y_i \multimap U_i$ and $G_i : X \times Y \times Y_i \multimap Z_i$ be multivalued maps with nonempty values and C_i and D_i be nonempty subsets of U_i and Z_i, respectively. If H_i and A_i are defined as follows:

$$H_i = \{y_i \in Y_i : F_i(u, y_i) \cap (-C_i) = \emptyset\}$$

and

$$A_i(y) = \{z_i \in Y_i : G_i(u, y, z_i) \cap (-D_i \setminus \{\theta_{Z_i}\}) \neq \emptyset\},$$

then HIDS will reduce to the following problem (P_2), which is an abstract equilibrium problem:

(P_2) Find $v = (v)_{i \in I} \in Y$ such that $F_i(u, v_i) \cap (-C_i) = \emptyset$ and $G_i(u, v, y_i) \cap (-D_i \setminus \{\theta_{Z_i}\}) = \emptyset$ for all $y_i \in T_i(v)$ and for all $i \in I$.

Example (C) For each $i \in I$, let $F_i : Y_i \multimap Y_i$ and $G_i : Y \times Y_i \multimap Y$ be multivalued maps. If H_i and A_i are defined as follows:

$$H_i = \{y_i \in Y_i : y_i \in F_i(y_i)\}$$

and

$$A_i(y) = \{z_i \in Y_i : \ y \notin G_i(y, z_i)\},$$

then HIDS will reduce to the following fixed point problem (P_3).

(P_3) Find $v = (v)_{i \in I} \in Y$ such that $v_i \in F_i(v_i)$ and $v \in G_i(v, y_i)$ for all $y_i \in T_i(v)$ for all $i \in I$;

Example (D) For each $i \in I$, let $F_i : X \times Y_i \to \mathbb{R}$ and $G_i : X \times Y \times Y_i \to \mathbb{R}$ be functions. If H_i and A_i are defined as following:

$$H_i = \{y_i \in Y_i : F_i(u, y_i) \leq 0\}$$

and

$$A_i(y) = \{z_i \in Y_i : \ G_i(u, y, z_i) \leq 0\},$$

then HIDS will reduce to the following system of hybrid scalar equilibrium problem (P_4):

(P_4) Find $v = (v)_{i \in I} \in Y$ such that $F_i(u, v_i) \leq 0$ and $G_i(u, v, y_i) > 0$ for all $y_i \in T_i(v)$ and for all $i \in I$;

Example (E) Let X be a nonempty Hausdorff t.v.s., $f : X \to (-\infty, \infty]$ and $p : X \times X \to (-\infty, \infty]$ be functions, $\varepsilon > 0$ and $u \in X$ be given. If H and A are defined as follows:

$$H = \{x \in X : \varepsilon p(u, x) \leq f(u) - f(x)\}$$

and

$$A(x) = \{y \in X : \ \varepsilon p(x, y) < f(x) - f(y)\},$$

then HIDS will reduce to Lin and Du's variant of Ekeland's variational principle in t.v.s.; see [15,17] (say problem (P_5)):

(P_5) Find $v \in X$ such that
(a) $\varepsilon p(u, v) \leq f(u) - f(v)$;
(b) $\varepsilon p(v, x) \geq f(v) - f(x)$ for all $x \in X$.

The paper is divided into five sections. In Sect. 3, we first establish the existence theorems of the solution of HIDS, from which we study mixed types of systems of generalized quasi-variational inclusions and disclusions problems and give some applications to the existence theorems of feasible points for various mathematical programs with variational constraints or equilibrium constraints, the existence theorems of system of vector saddle point and system of minimax theorem in Sect. 4.

2 Preliminaries

Throughout this paper we denote by $\mathbb{R}$ and $\mathbb{N}$ the set of real numbers and the set of positive integers, respectively. Let A and B be nonempty sets. A multivalued map $T : A \multimap B$ is a function from A to the power set 2^B of B. We denote $T(A) = \bigcup\{T(x) : x \in A\}$ and let $T^- : B \multimap A$ be defined by the condition that $x \in T^-(y)$ if and only if $y \in T(x)$. Recall that a nonempty subset C of a linear space X with its zero vector θ_X is called a *convex cone* if $C + C \subseteq C$ and $\lambda C \subseteq C$ for all $\lambda \geq 0$. A convex cone C in X is pointed if $C \cap (-C) = \{\theta_X\}$. Let X and Y be t.s. A multivalued map $T : X \multimap Y$ is said to be (1) upper semicontinuous (u.s.c., for short) at $x \in X$ if for every open set V in Y with $T(x) \subset V$, there exists an open neighborhood $U(x)$ of x such that $T(x') \subset V$ for all $x' \in U(x)$; (2) lower semicontinuous (l.s.c., for short) at $x \in X$ if for every open set V in Y with $T(x) \bigcap V \neq \emptyset$, there exists an open neighborhood $U(x)$ of x such that $T(x') \bigcap V \neq \emptyset$ for all $x' \in U(x)$; (3) u.s.c. (resp. l.s.c.) on X if T is u.s.c. (resp. l.s.c.) at every point of X ; (4) closed if $GrT = \{(x, y) : x \in X, y \in T(x)\}$, the graph of T, is closed in $X \times Y$; (5) compact if there exists a compact set K such that $T(X) \subseteq K$.

Let Z be a real t.v.s. with its zero vector θ_Z, D a proper convex cone in Z and $A \subseteq Z$. A point $\bar{y} \in A$ is called a vectorial minimal point of A with respect to D if for any $y \in A$, $y - \bar{y} \notin -D \setminus \{\theta_Z\}$. The set of vectorial minimal point of A is denoted by $\mathrm{Min}_D A$. The convex hull of A and the closure of A are denoted by coA and clA, respectively.

Definition 2.1 Let X be a nonempty convex subset of a vector space E, Y a nonempty convex subset of a vector space V and Z a real t.v.s. Let $F : X \times Y \multimap Z$ and $C : X \multimap Z$ be multivalued maps such that for each $x \in X$, $C(x)$ is a nonempty closed convex cone. For each fixed $x \in X$, $y \multimap F(x, y)$ is called $C(x)$-*quasiconvex* (resp. $C(x)$-*quasiconvex-like*) if for any $y_1, y_2 \in Y$ and $\lambda \in [0, 1]$, we have either

$$F(x, y_1) \subseteq F(x, \lambda y_1 + (1 - \lambda)y_2) + C(x)$$
$$(\text{resp. } F(x, \lambda y_1 + (1 - \lambda)y_2) \subseteq F(x, y_1) - C(x))$$

or

$$F(x, y_2) \subseteq F(x, \lambda y_1 + (1 - \lambda)y_2) + C(x)$$
$$(\text{resp. } F(x, \lambda y_1 + (1 - \lambda)y_2) \subseteq F(x, y_2) - C(x)).$$

The following Lemmas are crucial in this paper.

Lemma 2.1 [3,23] *Let X and Y be Hausdorff topological spaces and $T : X \multimap Y$ a multi-valued map. Then T is l.s.c. at $x \in X$ if and only if for any $y \in T(x)$ and for any net $\{x_\alpha\}$ in X converging to x , there exists a subnet $\{x_{\phi(\lambda)}\}_{\lambda \in \Lambda}$ of $\{x_\alpha\}$ and a net $\{y_\lambda\}_{\lambda \in \Lambda}$ with $y_\lambda \to y$ such that $y_\lambda \in T(x_{\phi(\lambda)})$ for all $\lambda \in \Lambda$.*

Lemma 2.2 [20] *Let Z be a Hausdorff t.v.s. and C be a closed convex cone in Z. If A is a nonempty compact subset of Z, then* $\mathrm{Min}_C A \neq \emptyset$.

Lemma 2.3 [4] *Let X and Y be Hausdorff topological spaces and $T : X \multimap Y$ a multivalued map.*

(i) *If T is an u.s.c. multivalued map with closed values, then T is closed;*
(ii) *If Y is a compact space and T is closed, then T is u.s.c.;*
(iii) *If X is compact and T is an u.s.c. multivalued map with compact values, then $T(X)$ is compact.*

3 Existence theorems of the solution of HIDS

The following result is needed in this paper.

Theorem 3.1 [7,15,17] *Let I be any index set. Let $\{X_i\}_{i\in I}$ be a family of nonempty convex subsets, where each X_i is contained in a Hausdorff t.v.s. E_i. For each $i \in I$, let $S_i : X = \prod_{i\in I} X_i \multimap X_i$ be a multivalued map such that*

(i) *for each $x = (x_i)_{i\in I} \in X$, $x_i \notin coS_i(x)$;*
(ii) *for each $y_i \in X_i$, $S_i^-(y_i)$ is open in X;*
(iii) *there exist a nonempty compact subset K of X and a nonempty compact convex subset M_i of X_i for all $i \in I$ such that for each $x \in X \setminus K$, there exists $j \in I$ such that $M_j \cap S_j(x) \neq \emptyset$.*

Then there exists $\bar{x} \in X$ such that $S_i(\bar{x}) = \emptyset$ for all $i \in I$.

In this section, we first establish an existence theorem of the solution of HIDS which is one of the main results of this paper.

Theorem 3.2 *Let I be any index set. For each $i \in I$, let Y_i be a nonempty closed convex subset of a Hausdorff t.v.s. V_i. Let X be a nonempty subset of a topological space E and $Y = \prod_{i\in I} Y_i$. For each $i \in I$, let $T_i : Y \multimap Y_i$ be a multivalued map with nonempty values. Let $u \in X$. For each $i \in I$, let H_i be a nonempty closed subset of Y_i and let $A_i : Y \multimap Y_i$ be a multivalued map. For each $i \in I$, suppose that the following conditions are satisfied:*

(i) *for each $y = (y_i)_{i\in I} \in Y$, $y_i \notin A_i(y)$;*
(ii) *for each $y \in Y$, $coT_i(y) \subseteq H_i$ and $A_i(y)$ is convex;*
(iii) *for each $z_i \in Y_i$, $T_i^-(z_i)$ and $A_i^-(z_i)$ are open in Y;*
(iv) *there exist a nonempty compact subset K of Y and a nonempty compact convex subset M_i of Y_i for each $i \in I$ such that for each $y \in Y \backslash K$ there exist $j \in I$ and $z_j \in M_j \cap T_j(y) \cap A_j(y)$.*

Then there exists $v = (v_i)_{i\in I} \in Y$ such that for each $i \in I$, $v_i \in H_i$ and $y_i \notin A_i(v)$ for all $y_i \in T_i(v)$.

Proof For each $i \in I$, let $W_i = H_i \times \prod_{j\neq i} Y_j$. Then W_i is a nonempty closed subset of Y for all $i \in I$. Define a multivalued map $\varphi_i : Y \multimap Y_i$ by

$$\varphi_i(y) = \begin{cases} T_i(y) \cap A_i(y), & \text{if } y \in W_i \\ T_i(y), & \text{if } y \in Y \backslash W_i. \end{cases}$$

Then for each $i \in I$, $y_i \notin co\varphi_i(y)$ for all $y = (y_i)_{i\in I} \in Y$. Indeed, for each $i \in I$, if $y \in W_i$, then $\varphi_i(y) = T_i(y) \cap A_i(y) \subseteq A_i(y)$. By (ii), we have $co\varphi_i(y) \subseteq A_i(y)$, which implies $y_i \notin co\varphi_i(y)$ from (i). On the other hand, if $y \in Y\backslash W_i$, then $y_i \notin H_i$. By (ii) again, we have $y_i \notin coT_i(y) = co\varphi_i(y)$. Hence for each $i \in I$, $y_i \notin co\varphi_i(y)$ for all $y = (y_i)_{i\in I} \in Y$. It is easy to see that for each $i \in I$ and $z_i \in Y_i$,

$$\varphi_i^-(z_i) = \left[T_i^-(z_i) \cap A_i^-(z_i)\right] \cup \left[(Y\backslash W_i) \cap T_i^-(z_i)\right].$$

Thus, from our hypothesis, $\varphi_i^-(z_i)$ is open in Y for each $(i, z_i) \in I \times Y_i$. By (iv), there exist a nonempty compact subset K of Y and a nonempty compact convex subset M_i of Y_i for each $i \in I$ such that for each $y \in Y\backslash K$ there exist $j \in I$, such that $M_j \cap \varphi_j(y) \neq \emptyset$. Applying Theorem 3.1, there exists $v \in Y$ such that $\varphi_i(v) = \emptyset$ for all $i \in I$. If $v \notin W_i$, then $\emptyset \neq T_i(v) = \varphi_i(v) = \emptyset$, which leads to a contradiction. Hence $v \in W_i$ and $y_i \notin A_i(v)$ for all $y_i \in T_i(v)$. Therefore there exists $v = (v_i)_{i\in I} \in Y$ such that for each $i \in I$, $v_i \in H_i$ and $y_i \notin A_i(v)$ for all $y_i \in T_i(v)$. □

Remark 3.1 Theorem 3.2 gives sufficient conditions for the existence of the solution of HIDS.

Below, unless otherwise specified in this section, we shall assume that I, Y_i, V_i, X, E, Y and T_i are the same as in Theorem 3.2 and U_i and Z_i real t.v.s. with zero vectors θ_{U_i} and θ_{Z_i}, respectively.

Theorem 3.3 *For each $i \in I$, let $N_i : X \times Y_i \multimap U_i$, $F_i : X \times Y_i \multimap U_i$, $G_i : X \times Y \times Y_i \multimap Z_i$ multivalued maps with nonempty values and O_i a nonempty open set in Z_i. Let $u \in X$ be given. For each $i \in I$, let $H_i = \{y_i \in Y_i : \theta_{U_i} \in F_i(u, y_i) + N_i(u, y_i)\}$ (resp. $H_i = \{y_i \in Y_i : \theta_{U_i} \notin F_i(u, y_i) + N_i(u, y_i)\}$). For each $i \in I$, suppose that*

(i) *H_i is a nonempty closed subset of Y_i;*
(ii) *for each $y = (y_i)_{i\in I} \in Y$, $\theta_{Z_i} \notin G_i(u, y, y_i) + O_i$;*
(iii) *for each $y \in Y$, $coT_i(y) \subseteq H_i$ and for each $z_i \in Y_i$, $T_i^-(z_i)$ is open in Y;*
(iv) *for each $y \in Y$, $G_i(u, y, \cdot)$ is $\{\theta_{Z_i}\}$ -quasiconvex and for each $z_i \in Y_i$, $G_i(u, \cdot, z_i)$ is l.s.c.;*
(v) *there exist a nonempty compact subset K of Y and a nonempty compact convex subset M_i of Y_i for each $i \in I$ such that for each $y \in Y\backslash K$ there exist $j \in I$ and $z_j \in M_j \cap T_j(y)$ such that $\theta_{Z_j} \in G_j(u, y, z_j) + O_j$.*

Then there exists $v = (v_i)_{i\in I} \in Y$ such that for each $i \in I$,

$$\theta_{U_i} \in F_i(u, v_i) + N_i(u, v_i) \text{ (resp. } \theta_{U_i} \notin F_i(u, v_i) + N_i(u, v_i))$$

and

$$\theta_{Z_i} \notin G_i(u, v, y_i) + O_i$$

for all $y_i \in T_i(v)$.

Proof For each $i \in I$, let $A_i : Y \multimap Y_i$ be defined by

$$A_i(y) = \{z_i \in Y_i : \theta_{Z_i} \in G_i(u, y, z_i) + O_i\}.$$

We claim that for each $(i, z_i) \in I \times Y_i$, $A_i^-(z_i)$ is open in Y. Let $y \in cl(Y\backslash A_i^-(z_i))$. Then there exists a net $\{y_\alpha\}_{\alpha\in\Lambda}$ in $Y\backslash A_i^-(z_i)$ such that $y_\alpha \to y$. Thus we have $\theta_{Z_i} \notin G_i(u, y_\alpha, z_i) + O_i$ or $G_i(u, y_\alpha, z_i) \subseteq Z_i\backslash O_i$. By the closedness of Y, $y \in Y$. Also, we obtain $G_i(u, y, z_i) \subseteq Z_i\backslash O_i$. Indeed, for any $w \in G_i(u, y, z_i)$, since $G_i(u, \cdot, z_i)$ is l.s.c. at y and $y_\alpha \to y$, by Lemma 2.1, there exists a net $\{w_\alpha\}$ with $w_\alpha \to w$ such that $w_\alpha \in G_i(u, y_\alpha, z_i) \subseteq Z_i\backslash O_i$.

Since $Z_i\backslash O_i$ is closed, we have $w \in Z_i\backslash O_i$ and hence $G_i(u, y, z_i) \subseteq Z_i\backslash O_i$. Therefore, $y \in Y\backslash A_i^-(z_i)$ and hence $A_i^-(z_i)$ is open in Y. Next, we show that for each $(i, y) \in I \times Y$, $A_i(y)$ is convex. Let $a_i, b_i \in A_i(y)$. Then $\theta_{Z_i} \in G_i(u, y, a_i)+O_i$ and $\theta_{Z_i} \in G_i(u, y, b_i)+O_i$. By the convexity of Y_i, $e_i^{(\lambda)} := \lambda a_i + (1-\lambda)b_i \in Y_i$, for all $\lambda \in [0, 1]$. Suppose to the contrary that there exists $\lambda_0 \in (0, 1)$ such that $\theta_{Z_i} \notin G_i\left(u, y, e_i^{(\lambda_0)}\right) + O_i$. By the $\{\theta_{Z_i}\}$-quasiconvexity of $G_i(u, y, \cdot)$, either

$$\theta_{Z_i} \in G_i(u, y, a_i) + O_i \subseteq G_i\left(u, y, e_i^{(\lambda_0)}\right) + O_i$$

or

$$\theta_{Z_i} \in G_i(u, y, b_i) + O_i \subseteq G_i\left(u, y, e_i^{(\lambda_0)}\right) + O_i,$$

which leads to a contradiction. Hence for each $(i, y) \in I \times Y$, $A_i(y)$ is convex. Therefore, all the conditions of Theorem 3.2 are satisfied and the conclusion follows from Theorem 3.2 □

Remark 3.2 Let X and Y be t.v.s., U a real t.v.s. with its zero vector θ and $u \in X$.

(a) Let $C \neq \{\theta\}$ be a point convex cone in Y and $F : X \times Y \multimap U$ a multivalued map defined by $F(u, y) = \theta$ for all $y \in Y$. Then $H_1 = \{y \in Y : \theta \in F(u, y)\} = Y$ and $H_2 = \{y \in Y : F(u, y) \subseteq C\} = Y$ are closed;
(b) Let $C \neq \{\theta\}$ be a point convex cone in Y. If a multivalued map $F : X \times Y \multimap U$ is defined by $F(u, y) = C\backslash\{\theta\}$, then $H_3 = \{y \in Y : \theta \notin F(u, y)\} = Y$ and $H_4 = \{y \in Y : F(u, y) \not\subseteq C\} = Y$ are closed;
(c) Let $F : X \times Y \multimap U$ be a multivalued map with nonempty values and $C : X \times Y \multimap U$ be a multivalued map with nonempty values such that $y \multimap C(u, y)$ is closed. If there exists $w = w(u) \in Y$ such that $F(u, w) \subseteq C(u, w)$ and $F(u, \cdot)$ is l.s.c., then $H = \{y \in Y : F(u, y) \subseteq C(u, y)\}$ is a nonempty closed subset of Y;
(d) Let $F : X \times Y \multimap U$ be a multivalued map with nonempty values such that there exists $w = w(u) \in Y$ such that $\theta \in F(u, w)$ and the map $y \multimap F(u, y)$ is closed. Then $H = \{y \in Y : \theta \in F(u, y)\}$ is a nonempty closed subset of Y.

Subsequently, we establish some existence theorems of systems of generalized vector quasiequilibrium problems.

Theorem 3.4 *For each $i \in I$, let $C_i : X \times Y_i \multimap U_i$, $D_i : X \times Y \multimap Z_i$, $F_i : X \times Y_i \multimap U_i$, $G_i : X \times Y \times Y_i \multimap Z_i$ and $T_i : Y \multimap Y_i$ be multivalued maps with nonempty values. Let $u \in X$. For each $i \in I$, let $H_i = \{y_i \in Y_i : F_i(u, y_i) \cap (-C_i(u, y_i)) = \emptyset\}$ (resp. $H_i = \{y_i \in Y_i : F_i(u, y_i) \cap (-C_i(u, y_i)) \neq \emptyset\}$). For each $i \in I$, suppose that*

(i) *H_i is a nonempty closed subset of Y_i;*
(ii) *for each $y = (y_i)_{i\in I} \in Y$, $G_i(u, y, y_i) \cap (-D_i(u, y)\backslash\{\theta_{Z_i}\}) = \emptyset$;*
(iii) *$R_i : Y \multimap Z_i$ is closed, where $R_i(y) = Z_i\backslash(-D_i(u, y)\backslash\{\theta_{Z_i}\})$ for $y \in Y$;*
(iv) *for each $y \in Y$, $coT_i(y) \subseteq H_i$ and for each $z_i \in Y_i$, $T_i^-(z_i)$ is open in Y;*
(v) *for each $y \in Y$, $D_i(u, y)$ is a nonempty convex cone and $D_i(u, y) \neq \{\theta_{Z_i}\}$;*
(vi) *for each $y \in Y$, $G_i(u, y, \cdot)$ is $D_i(u, y)$ -quasiconvex and for each $z_i \in Y_i$, $G_i(u, \cdot, z_i)$ is l.s.c.;*
(vii) *there exist a nonempty compact subset K of Y and a nonempty compact convex subset M_i of Y_i for each $i \in I$ such that for each $y \in Y\backslash K$ there exist $j \in I$ and $z_j \in M_j \cap T_j(y)$ such that $G_j(u, y, z_j) \cap (-D_j(u, y)\backslash\{\theta_{Z_j}\}) \neq \emptyset$.*

Then there exists $v = (v_i)_{i \in I} \in Y$ *such that for each* $i \in I$,

$$F_i(u, v_i) \cap (-C_i(u, v_i)) = \emptyset \text{ (resp. } F_i(u, v_i) \cap (-C_i(u, v_i)) \neq \emptyset)$$

and

$$G_i(u, v, y_i) \cap (-D_i(u, v) \backslash \{\theta_{Z_i}\}) = \emptyset$$

for all $y_i \in T_i(v)$.

Proof For each $i \in I$, let $A_i : Y \multimap Y_i$ be defined by

$$A_i(y) = \{z_i \in Y_i : G_i(u, y, z_i) \cap (-D_i(u, y) \backslash \{\theta_{Z_i}\}) \neq \emptyset\}.$$

We first show that for each $(i, z_i) \in I \times Y_i$, $A_i^-(z_i)$ is open in Y. Let $y \in cl(Y \backslash A_i^-(z_i))$. Then there exists a net $\{y_\alpha\}_{\alpha \in \Lambda}$ in $Y \backslash A_i^-(z_i)$ such that $y_\alpha \to y$. Thus we have $G_i(u, y_\alpha, z_i) \cap (-D_i(u, y_\alpha) \backslash \{\theta_i\}) = \emptyset$ or $G_i(u, y_\alpha, z_i) \subseteq R_i(y_\alpha)$. By the closedness of Y, $y \in Y$. Also, we obtain $G_i(u, y, z_i) \subseteq R_i(y)$. Indeed, for any $w \in G_i(u, y, z_i)$, since $G_i(u, \cdot, z_i)$ is l.s.c. at y and $y_\alpha \to y$, by Lemma 2.1, there exists a net $\{w_\alpha\}$ with $w_\alpha \to w$ such that $w_\alpha \in G_i(u, y_\alpha, z_i) \subseteq R_i(y_\alpha)$. Since R_i is closed, we have $w \in R_i(y)$. Thus $G_i(u, y, z_i) \subseteq R_i(y)$. Therefore $y \in Y \backslash A_i^-(z_i)$ and hence $A_i^-(z_i)$ is open in Y. Next, we claim that for each $(i, y) \in I \times Y$, $A_i(y)$ is convex. Let $a_i, b_i \in A_i(y)$. Then $G_i(u, y, a_i) \cap (-D_i(u, y) \backslash \{\theta_{Z_i}\}) \neq \emptyset$ and $G_i(u, y, b_i) \cap (-D_i(u, y) \backslash \{\theta_{Z_i}\}) \neq \emptyset$. By the convexity of Y_i, $e_i^{(\lambda)} := \lambda a_i + (1 - \lambda) b_i \in Y_i$, for all $\lambda \in [0, 1]$. Suppose to the contrary that there exists $\lambda_0 \in (0, 1)$ such that $G_i\left(u, y, e_i^{(\lambda_0)}\right) \cap (-D_i(u, y) \backslash \{\theta_{Z_i}\}) = \emptyset$. By the $D_i(u, y)$-quasiconvexity of $G_i(u, y, \cdot)$, either

$$\begin{aligned} & G_i(u, y, a_i) \cap (-D_i(u, y) \backslash \{\theta_{Z_i}\}) \\ & \quad \subseteq \left[G_i\left(u, y, e_i^{(\lambda_0)}\right) + D_i(u, y)\right] \cap (-D_i(u, y) \backslash \{\theta_{Z_i}\}) = \emptyset \end{aligned}$$

or

$$\begin{aligned} & G_i(u, y, b_i) \cap (-D_i(u, y) \backslash \{\theta_{Z_i}\}) \\ & \quad \subseteq \left[G_i\left(u, y, e_i^{(\lambda_0)}\right) + D_i(u, y)\right] \cap (-D_i(u, y) \backslash \{\theta_{Z_i}\}) = \emptyset. \end{aligned}$$

This leads to a contradiction. Hence for each $(i, y) \in I \times Y$, $A_i(y)$ is convex. For each $i \in I$, let $F_i^{'}(x, y_i) = F_i(x, y_i) + C_i(x, y_i)$ and $G_i^{'}(x, y, z_i) = G_i(x, y, z_i) + (D_i(x, y) \backslash \{\theta_{Z_i}\})$. Therefore all the conditions of Theorem 3.3 are satisfied and the conclusion follows from Theorem 3.3 □

Following a similar argument as in Theorem 3.4, we have the following result.

Theorem 3.5 *In Theorem 3.4 , if* $H_i = \{y_i \in Y_i : F_i(u, y_i) \cap (-int C_i(u, y_i)) = \emptyset\}$ *(resp.* $H_i = \{y_i \in Y_i : F_i(u, y_i) \cap (-int C_i(u, y_i)) \neq \emptyset\}$*) and conditions (ii), (iii) and (vii) are replaced by (ii)*$_a$*, (iii)*$_a$ *and (vii)*$_a$ *respectively, where*

(ii)$_a$ *for each* $y = (y_i)_{i \in I} \in Y$, $G_i(u, y, y_i) \cap (-int D_i(u, y)) = \emptyset$;

(iii)$_a$ $R_i : Y \multimap Z_i$ *is closed, where* $R_i(y) = Z_i \backslash (-int D_i(u, y))$ *for* $y \in Y$;

(vii)$_a$ *there exist a nonempty compact subset* K *of* Y *and a nonempty compact convex subset* M_i *of* Y_i *for each* $i \in I$ *such that for each* $y \in Y \backslash K$ *there exist* $j \in I$ *and* $z_j \in M_j \cap T_j(y)$ *such that* $G_j(u, y, z_j) \cap (-int D_j(u, y)) \neq \emptyset$.

Then there exists $v = (v_i)_{i\in I} \in Y$ *such that for each* $i \in I$,

$$F_i(u, v_i) \cap (-intC_i(u, v_i)) = \emptyset \text{ (resp. } F_i(u, v_i) \cap (-intC_i(u, v_i)) \neq \emptyset)$$

and

$$G_i(u, v, y_i) \cap (-intD_i(u, v)) = \emptyset$$

for all $y_i \in T_i(v)$.

4 Some applications

By using Theorem 3.3, we can prove an existence theorem of system of vector saddle point.

Theorem 4.1 *Let* $n \in \mathbb{N}$ *and* $I = \{1, 2, \ldots, n\}$. *For each* $i \in I$, *let* X_i *be a nonempty closed convex subset of a Hausdorff t.v.s.* V_i, Z_i *real t.v.s. with zero vector* θ_{Z_i} *and* $L_i : X_i \times X_i \to Z_i$ *a map. Let* $X = \prod_{i\in I} X_i$ *and let* $u = (u_1, \ldots, u_n) \in X$ *be given. For each* $i \in I$, *let* C_i *be a nonempty proper convex cone in* Z_i *such that* $Z_i \backslash (-C_i \backslash \{\theta_{Z_i}\})$ *is closed. Suppose that*

(i) *for each* $x_i \in X_i$, $y_i \to L_i(x_i, y_i)$ *is continuous and* $\{\theta_{Z_i}\}$*-quasiconvex;*
(ii) *for each* $y_i \in X_i$, $x_i \to L_i(x_i, y_i)$ *is* $\{\theta_{Z_i}\}$*-quasiconvex-like;*
(iii) $[L_i(u_i, y_i) - L_i(y_i, y_i)] \notin (-C_i \backslash \{\theta_{Z_i}\})$ *for all* $y_i \in X_i$;
(iv) *Suppose that there exist a nonempty compact subset* K *of* X *and a nonempty compact convex subset* M_i *of* X_i *for each* $i \in I$ *such that for each* $y = (y_i)_{i\in I} \in X\backslash K$ *there exist* $j \in I$ *and* $z_j \in M_j$ *such that* $[L_j(u_j, z_j) - L_j(u_j, y_j)] \notin (-C_j \backslash \{\theta_{Z_j}\})$ *and* $[L_j(u_j, y_j) - L_j(z_j, y_j)] \notin (-C_j \backslash \{\theta_j\})$.

Then there exists $v = (v_i)_{i\in I} \in X$ *such that for each* $i \in I$,

$$[L_i(u_i, y_i) - L_i(u_i, v_i)] \notin (-C_i \backslash \{\theta_{Z_i}\})$$

and

$$[L_i(u_i, v_i) - L_i(y_i, v_i)] \notin (-C_i \backslash \{\theta_{Z_i}\})$$

for all $y_i \in X_i$.

Proof Put $X'_{2k-1} = X'_{2k} = X_k, C'_{2k-1} = C'_{2k} = C_k, L'_{2k-1} = L'_{2k} = L_k$ and $u'_{2k-1} = u'_{2k} = u_k$ for each $k \in I$. Let $J = \{1, 2, \ldots, 2n\}$. For each $i \in J$, let $Y'_i = X'_i$ and $X' = Y' = \prod_{i\in J} X'_i$. Then $u' := (u'_1, u'_2, \ldots, u'_{2n}) \in X'$. For each $i \in J$, define $F_i : X' \times X'_i \to Z_i$ by $F_i(x, y_i) = C'_i \backslash \{\theta_{Z_i}\}$. Thus

$$H_i = \{x'_i \in X'_i : \theta_{Z_i} \notin F_i(u', x'_i)\} = X'_i.$$

For each $i \in J$, let $T_i : X' \multimap X'_i$ by

$$T_i(y) = H_i = X'_i \text{ for each } y \in X' \iff T_i^-(z_i) = X'_i \text{ for each } z_i \in X'_i.$$

Then for each $(i, y) \in J \times X'$, $coT_i(y) \subseteq H_i = X'_i$ and for each $(i, z_i) \in J \times X'_i$, $T_i^-(z_i)$ is open in X'. For each $i \in J$, define $f_i : X' \times X' \times X'_i \to Z_i$ by

$$f_i(x, y, z_i) = L'_i(x_i, y_i) - L'_i(z_i, y_i),$$

and define $g_i : X' \times X' \times X'_i \to Z_i$ by

$$g_i(x, y, z_i) = L'_i(x_i, z_i) - L'_i(x_i, y_i).$$

For each $k \in \mathbb{N}$, let $G_{2k-1} = f_k$ and $G_{2k} = g_k$. Then for each $i \in J$, $G_i : X' \times X' \times X'_i \to Z_i$ is a map such that for each $y = (y_i)_{i \in I} \in X'$, $G_i(u', y, y_i) \notin (-C_i \backslash \{\theta_{Z_i}\})$ from (iii). For each $i \in J$, we claim that $G_i(u', y, \cdot)$ is C'_i-quasiconvex for each $y \in X'$. Let $z_i^1, z_i^2 \in X'_i$ and $\lambda \in [0, 1]$. By the $\{\theta_{Z_i}\}$-quasiconvexity of $L'_i(x_i, \cdot)$, either

$$L'_i\left(x_i, \lambda z_i^1 + (1-\lambda) z_i^2\right) = L'_i\left(x_i, z_i^1\right)$$

or

$$L'_i\left(x_i, \lambda z_i^1 + (1-\lambda) z_i^2\right) = L'_i\left(x_i, z_i^2\right).$$

By (ii), we have either

$$L'_i\left(\lambda z_i^1 + (1-\lambda) z_i^2, y_i\right) = L'_i\left(z_i^1, y_i\right)$$

or

$$L'_i\left(\lambda z_i^1 + (1-\lambda) z_i^2, y_i\right) = L'_i\left(z_i^2, y_i\right).$$

Hence either

$$\begin{aligned} f_i\left(x, y, z_i^1\right) &= L'_i(x_i, y_i) - L'_i\left(z_i^1, y_i\right) \\ &= L'_i(x_i, y_i) - L'_i\left(\lambda z_i^1 + (1-\lambda) z_i^2, y_i\right) \\ &\subseteq f_i\left(x, y, \lambda z_i^1 + (1-\lambda) z_i^2\right) + C'_i \end{aligned}$$

or

$$f_i\left(x, y, z_i^2\right) \subseteq f_i\left(x, y, \lambda z_i^1 + (1-\lambda) z_i^2\right) + C'_i.$$

Also, we have either

$$g_i\left(x, y, z_i^1\right) \subseteq g_i\left(x, y, \lambda z_i^1 + (1-\lambda) z_i^2\right) + C'_i$$

or

$$g_i\left(x, y, z_i^2\right) \subseteq g_i\left(x, y, \lambda z_i^1 + (1-\lambda) z_i^2\right) + C'_i.$$

So for each $i \in J$, $G_i(u', y, \cdot)$ is C'_i -quasiconvex for each $y \in X'$. Since for each $x_i \in X_i$, $y_i \to L_i(x_i, y_i)$ is continuous, we have $f_i(u', y, z_i) = L'_i\left(u'_i, y_i\right) - L'_i(z_i, y_i)$ and $g_i(u', y, z_i) = L'_i\left(u'_i, z_i\right) - L'_i\left(u'_i, y_i\right)$ are l.s.c. for each $z_i \in X'_i$. Hence for each $i \in J$, $G_i(u', \cdot, z_i)$ is l.s.c. for each $z_i \in X'_i$. By Theorem 3.3, there exists $v = (v_i)_{i \in J} \in X'$ such that for each $i \in J$, $G_i(u', v, y_i) \notin (-C_i \backslash \{\theta_{Z_i}\})$ for all $y_i \in X'_i$, which is equivalent with for each $i \in I$,

$$[L_i(u_i, y_i) - L_i(u_i, v_i)] \notin (-C_i \backslash \{\theta_{Z_i}\})$$

and

$$[L_i(u_i, v_i) - L_i(y_i, v_i)] \notin (-C_i \backslash \{\theta_{Z_i}\})$$

for all $y_i \in X_i$. □

The following new system of minimax theorem is immediate from Theorem 4.1

Theorem 4.2 (System of minimax theorem) *Let $n \in \mathbb{N}$ and $I = \{1, 2, \ldots, n\}$. For each $i \in I$, let X_i be a nonempty closed convex subset of a Hausdorff t.v.s. V_i and $L_i : X_i \times X_i \to \mathbb{R}$ a function. Let $X = \prod_{i \in I} X_i$ and let $u = (u_1, \ldots, u_n) \in X$ be given. Suppose that*

(i) *for each $x_i \in X_i$, $y_i \to L_i(x_i, y_i)$ is continuous and $\{0\}$-quasiconvex;*
(ii) *for each $y_i \in X_i$, $x_i \to L_i(x_i, y_i)$ is $\{0\}$-quasiconvex-like;*
(iii) *$L_i(u_i, y_i) \geq L_i(y_i, y_i)$ for all $y_i \in X_i$;*
(iv) *there exist a nonempty compact subset K of X and a nonempty compact convex subset M_i of X_i for each $i \in I$ such that for each $y = (y_i)_{i \in I} \in X \backslash K$ there exist $j \in I$ and $z_j \in M_j$ such that $L_j(u_j, z_j) \geq L_j(u_j, y_j)$ and $L_j(u_j, y_j) \geq L_j(z_j, y_j)$.*

Then for each $i \in I$, $\sup_{x_i \in X_i} \inf_{y_i \in X_i} L_i(x_i, y_i) = \inf_{y_i \in X_i} \sup_{x_i \in X_i} L_i(x_i, y_i)$.

Proof For each $i \in I$, let $C_i = [0, \infty)$. By Theorem 4.1, there exists $v = (v_i)_{i \in I} \in X$ such that for each $i \in I$,

$$L_i(u_i, y_i) \geq L_i(u_i, v_i)$$

and

$$L_i(u_i, v_i) \geq L_i(y_i, v_i)$$

for all $y_i \in X_i$. It follows that

$$\sup_{x_i \in X_i} \inf_{y_i \in X_i} L_i(x_i, y_i) \geq L_i(u_i, v_i) \geq \inf_{y_i \in X_i} \sup_{x_i \in X_i} L_i(x_i, y_i)$$

and hence

$$\sup_{x_i \in X_i} \inf_{y_i \in X_i} L_i(x_i, y_i) = \inf_{y_i \in X_i} \sup_{x_i \in X_i} L_i(x_i, y_i).$$

□

The following results are existence theorems of feasible points for mathematical programs with equilibrium constraints.

Theorem 4.3 *For each $i \in I$, let $f_i : X \times Y_i \to (-\infty, \infty]$ and $g_i : X \times Y \times Y_i \to (-\infty, \infty]$ be functions, $T_i : Y \multimap Y_i$ be a multivalued map with nonempty values, and let $H_i = \{y_i \in Y_i : f_i(u, y_i) \leq 0\}$, where $y = (y_i)_{i \in I} \in Y$. Let $u \in X$. For each $i \in I$, suppose that there exists $w_i \in Y_i$ such that $f_i(u, w_i) \leq 0$. For each $i \in I$, suppose that*

(i) *$f_i(u, \cdot)$ is l.s.c.;*
(ii) *for each $y = (y_i)_{i \in I} \in Y$, $g_i(u, y, y_i) \geq 0$;*
(iii) *for each $y \in Y$, $coT_i(y) \subseteq H_i$ and for each $z_i \in Y_i$, $T_i^-(z_i)$ is open in Y;*
(iv) *$g_i(u, \cdot, \cdot)$ is u.s.c. and for each $y \in Y$, $g_i(u, y, \cdot)$ is quasiconvex;*
(v) *there exist a nonempty compact subset K of Y and a nonempty compact convex subset M_i of Y_i for each $i \in I$ such that for each $y \in Y \backslash K$ there exist $j \in I$ and $z_j \in M_j \cap T_j(y)$ such that $g_j(u, y, z_j) < 0$.*

Let $h : X \times Y \multimap Z_0$ be a multivalued map such that $y \multimap h(u, y)$ is an u.s.c. multivalued map with nonempty compact values, where Z_0 is a real t.v.s. ordered by a proper closed convex cone C in Z_0 . Then there exists an optimal solution to the following problem $(\mathcal{P})$:

$$\begin{array}{c}\mathrm{Min}_C\, h(u, y)\\ \text{object to } y \in Y,\ \ f_i(u, y_i) \le 0 \text{ and } g_i(u, y, z_i) \ge 0\\ \text{for all } z_i \in T_i(y) \text{ and for all } i \in I.\end{array} \qquad (\mathcal{P})$$

Proof For each $i \in I$, let

$$N_i = \{y \in Y : f_i(u, y_i) \le 0 \text{ and } g_i(u, y, z_i) \ge 0 \text{ for all } z_i \in T_i(y)\}.$$

For each $i \in I$, let $y_i \in cl N_i$. Then there exists a net $\{y_i^\alpha\}_{\alpha\in\Lambda}$ in N_i such that $y_i^\alpha \to y_i$. Hence $f_i\left(u, y_i^\alpha\right) \le 0$ and $g_i\left(u, y_i^\alpha, z_i\right) \ge 0$ for all $z_i \in T_i\left(y_i^\alpha\right)$. Let $a_i \in T_i(y)$. Since $T_i^-(z_i)$ is open in Y for each $z_i \in Y_i$, T_i is l.s.c. Hence there exists a net $\{a_i^\alpha\}_{\alpha\in\Lambda}$ with $a_i^\alpha \to a_i$ such that $a_i^\alpha \in T_i\left(y_i^\alpha\right)$. So $f_i\left(u, y_i^\alpha\right) \le 0$ and $g_i\left(u, y_i^\alpha, a_i^\alpha\right) \ge 0$. By (i), we have $f_i(u, y_i) \le 0$. By (iv), we have $g_i(u, y_i, a_i) \ge 0$. Hence $y_i \in N_i$ and N_i is a closed set in Y. Let $N = \cap_{i\in I} N_i$. Then N is closed in Y. Applying Theorem 3.2, $N \ne \emptyset$. By (v), it is easy to see that $N \subseteq K$, where K is a nonempty compact subset of Y in condition (v). Hence N is a nonempty compact subset of Y. Since the map $y \multimap h(u, y)$ is an u.s.c. multivalued map with nonempty compact values, it follows from Lemma 2.3 that $h(u, N)$ is compact. Then by Lemma 2.2 that $\mathrm{Min}_C\, h(u, N) \ne \emptyset$. That is there exists a solution to the problem $(\mathcal{P})$. The proof is completed. □

Theorem 4.4 *In Theorem 4.3, if we assume that $h : X \times Y \to (-\infty, \infty]$ is a l.s.c. function, then there exists an optimal solution to the problem $(\mathcal{P})$ as in Theorem 4.3*

Proof Let N be the same as in the proof of Theorem 4.3 By the lower semicontinuity of h and the compactness of N, there exists $v \in N$ such that $h(u, v) = \min h(u, N)$. The proof is completed. □

Let X be a t.v.s. Recall that a function $p : X \times X \to (-\infty, \infty]$ is called a *quasi-distance* [17] on X if the following are satisfied:

$(QD1)$ $p(x, x) \ge 0$ for all $x \in X$;
$(QD2)$ $p(x, z) \le p(x, y) + p(y, z)$ for any $x, y, z \in X$;
$(QD3)$ for any $x \in X$, $p(x, \cdot)$ is convex and l.s.c.;
$(QD4)$ for any $y \in X$, $p(\cdot, y)$ is u.s.c.

Lin and Du's variant of Ekeland's variational principle [17] for quasi-distances in a Hausdorff t.v.s. can be easily given by Theorem 3.2

Theorem 4.5 ([17], Theorem 4.1) *Let X be a Hausdorff t.v.s. Let $f : X \to (-\infty, \infty]$ be a l.s.c. and convex function and $p : X \times X \to (-\infty, \infty]$ be a quasi-distance. Let $u \in X$ with $p(u, u) = 0$ and $\varepsilon > 0$. Suppose that there exist a nonempty compact subset K of X and a nonempty compact convex subset M of X such that for each $y \in X\backslash K$ there exists $z \in M$ such that $\varepsilon p(u, z) \le f(u) - f(z)$ and $\varepsilon p(y, z) < f(y) - f(z)$. Then there exists $v \in X$ such that*

(i) $\varepsilon p(u, v) \le f(u) - f(v)$;
(ii) $\varepsilon p(v, x) \ge f(v) - f(x)$ *for all* $x \in X$.

Proof Since p is a quasi-distance, εp is also a quasi-distance. Define H and $A : X \multimap X$ by

$$H = \{x \in X : \varepsilon p(u, x) \le f(u) - f(x)\}$$

and

$$A(x) = \{y \in X :\ \varepsilon p(x, y) < f(x) - f(y)\},$$

respectively, and let $T : X \multimap X$ be defined by

$$T(x) = H \quad \text{for all } x \in X$$

$$\Longleftrightarrow \quad T^{-}(z) = \begin{cases} X, & \text{if } z \in H \\ \emptyset, & \text{if } z \in X \setminus H. \end{cases}$$

It is not hard to verify that all the conditions of Theorem 3.2 are satisfied. Thus there exists $v \in X$ such that

(i) $\varepsilon p(u, v) \leq f(u) - f(v)$;
(ii) $\varepsilon p(v, x) \geq f(v) - f(x)$ for all $x \in H$.

For any $x \in X \backslash H$, since

$$\begin{aligned} \varepsilon[p(u, v) + p(v, x)] &\geq \varepsilon p(u, x) \\ &> f(u) - f(x) \\ &\geq \varepsilon p(u, v) + f(v) - f(x), \end{aligned}$$

it follows that $\varepsilon p(v, x) > f(v) - f(x)$ for all $x \in X \backslash H$. Therefore $\varepsilon p(v, x) \geq f(v) - f(x)$ for all $x \in X$. The proof is completed. □

5 Conclusions

In the present paper, we first introduce the new mathematical model about HIDS which contains several important problems (see Sect. 1) as special cases in the literatures. We establish sufficient conditions for the existence of the solution of HIDS and study mixed types of systems of generalized quasivariational inclusions and disclusions problems and systems of generalized vector quasiequilibrium problems. Some applications to the existence theorems of feasible points for various mathematical programs with variational constraints or equilibrium constraints, the existence theorems of system of vector saddle point and system of minimax theorem are also given. Our method would be useful to improve and generalize a number of other known results; see e.g., [1,2,6–11,13–15,17–21].

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